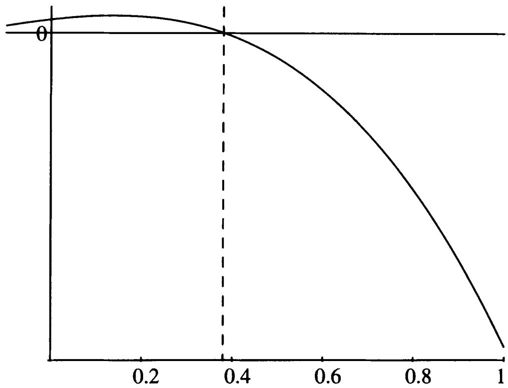

FIGURE 3: The graph of the quartic between 0 and 1. The coordinates of the point of intersection of the vertical and horizontal lines are (0.37995,0).

We believe that pupils should not be overprotected from the realities of mathematics and the above problem could be given to able students to teach them the mathematical facts of life.

PRITHWIJIT DE and DES MacHALE
School of Mathematical Sciences, University College Cork, Cork, Ireland
e-mails: *de.prithwijit@gmail.com; d.machale@ucc.ie*

91.63 An optimal inequality*

Let a, b, c be non-negative real numbers. Taking them in pairs and considering their arithmetic and geometric means yields three inequalities, of which a typical one is

$$\frac{b + c}{2} \geqslant \sqrt{bc}.$$

Adding the three inequalities, we get

$$a + b + c \geqslant \sqrt{bc} + \sqrt{ca} + \sqrt{ab}. \tag{1}$$

A quick computation shows that the maximum number ξ such that the inequality

$$a + b + c \geqslant \xi\left(\sqrt{bc} + \sqrt{ca} + \sqrt{ab}\right)$$

is true for all non-negative numbers a, b, c is $\xi = 1$. On the other hand, we have the following more general version of the above inequality:

Theorem 1: Let $a, b, c \geqslant 0$ and let $\alpha + \beta + \gamma = \pi$. Then

$$a + b + c \geqslant 2\sqrt{bc}\cos\alpha + 2\sqrt{ca}\cos\beta + 2\sqrt{ab}\cos\gamma. \tag{2}$$

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Proof: Since $\alpha + \beta + \gamma = \pi$, we have

$$\cos\alpha = -\cos(\beta + \gamma) = -\cos\beta\cos\gamma + \sin\beta\sin\gamma. \quad (3)$$

Thus we have

$$a + b + c - 2\sqrt{bc}\cos\alpha - 2\sqrt{ca}\cos\beta - 2\sqrt{ab}\cos\gamma$$

$$= \left(\sqrt{c} - \sqrt{a}\cos\beta - \sqrt{b}\cos\alpha\right)^2 + \left(\sqrt{a}\sin\beta - \sqrt{b}\sin\alpha\right)^2 \geqslant 0. \quad (4)$$

When $\alpha = \beta = \gamma = \frac{\pi}{3}$, (2) is the same as (1).

The main result of this short note is to prove that the above inequality is, in some sense, optimal.

We make the following definition: let (ξ, η, ζ) be a triple of non-negative numbers. We say that this triple is *feasible* if

$$a + b + c \geqslant 2\xi\sqrt{bc} + 2\eta\sqrt{ca} + 2\zeta\sqrt{ab} \qquad \text{for all real } a, b, c \geqslant 0. \quad (5)$$

We also say that the triple (ξ, η, ζ) is *optimal* if (a) it is feasible and (b) if (ξ', η', ζ') is any feasible triple such that $\xi' \geqslant \xi$, $\eta' \geqslant \eta$ and $\zeta' \geqslant \zeta$, then $\xi' = \xi$, $\eta' = \eta$ and $\zeta' = \zeta$.

Theorem 2: If the triple (ξ, η, ζ) is optimal, then there exist α, β, γ with $\alpha + \beta + \gamma = \pi$ such that $\xi = \cos\alpha$, $\eta = \cos\beta$ and $\zeta = \cos\gamma$.

Proof: Putting

$$\sqrt{c} = \eta\sqrt{a} + \xi\sqrt{b}$$

into (5), we have

$$\left(1 - \eta^2\right)a - 2(\zeta + \xi\eta)\sqrt{ab} + \left(1 - \xi^2\right)b \geqslant 0 \qquad \text{for all real } a, b \geqslant 0. \quad (6)$$

Taking $a = 0$ and $b = 1$, we get $1 - \xi^2 \geqslant 0$. Since $\xi \geqslant 0$, we have $\xi = \cos\alpha$ for some α in the range $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$. Similarly $\eta = \cos\beta$ in the range $0 \leqslant \beta \leqslant \frac{1}{2}\pi$. Moreover, from (6)

$$(\zeta + \xi\eta)^2 \leqslant \left(1 - \eta^2\right)\left(1 - \xi^2\right)$$

which is equivalent to

$$(\zeta + \cos\alpha\cos\beta)^2 \leqslant \sin^2\alpha\sin^2\beta.$$

But $0 \leqslant \zeta$, $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$ and $0 \leqslant \beta \leqslant \frac{1}{2}\pi$, so

$$0 \leqslant \zeta + \cos\alpha\cos\beta \leqslant \sin\alpha\sin\beta.$$

Taking $\gamma = \pi - \alpha - \beta$, we have $0 \leqslant \zeta \leqslant \cos\gamma$. Now $(\xi, \eta, \zeta) = (\cos\alpha, \cos\beta, \cos\zeta)$ is optimal and, from Theorem 1, $(\cos\alpha, \cos\beta, \cos\gamma)$ is feasible, so $\zeta = \cos\gamma$ as required.

As an application of Theorem 1, we can prove the famous Erdös-Mordell inequality which states that, if P is a point in the interior of a triangle ABC whose distances are p, q, r from the vertices of the triangle and x, y, z from its sides, then

$$p + q + r \geqslant 2(x + y + z).$$

To see this, let $\angle BPC = 2\alpha$, $\angle CPA = 2\beta$ and $\angle APB = 2\gamma$. Then $\alpha + \beta + \gamma = \pi$. With $PB = q$, $PC = r$ and $BC = a$, the distance, x, of P from BC is given by

$$ax = qr \sin 2\alpha, \tag{7}$$

while the cosine formula tells us that

$$a^2 = q^2 + r^2 - 2qr \cos 2\alpha. \tag{8}$$

As an exercise, readers are invited to deduce from (7) and (8) that $x \leqslant \sqrt{qr} \cos\alpha$. Similar inequalities hold for y and z. Using Theorem 1, we get

$$p + q + r \geqslant 2\left(\sqrt{qr}\cos\alpha + \sqrt{rp}\cos\beta + \sqrt{pq}\cos\gamma\right) \geqslant 2(x + y + z).$$

We are interested in the generalisation of the Erdös-Mordell inequality to n-polygons. The following inequality is true for any positive number n:

Let $a_1 + \ldots + a_n$ be n positive numbers. Then we have

$$a_1 + \ldots + a_n \geqslant \sqrt{a_1 a_2} + \ldots + \sqrt{a_{n-1} a_n} + \sqrt{a_n a_1}.$$

In general, we conjecture that

$$a_1, \ldots, a_n \geqslant \sec\frac{\pi}{n}\left(\sqrt{a_1 a_2}\cos\alpha_1 + \ldots + \sqrt{a_{n-1} a_n}\cos\alpha_{n-1} + \sqrt{a_n a_1}\cos\alpha_n\right)$$

for $\alpha_1 + \ldots + \alpha_n = \pi$.

It is not hard to see that the above inequality, if true, would give the generalisation of the Erdös-Mordell inequality to n-polygons.

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ZHIQIN LU

Department of Mathematics, University of California, Irvine, CA 92697, USA

e-mail: *zlu@math.uci.edu*

91.64 A touch of heuristics

It is my firm belief that heuristics, or the heuristic method, plays an important role in mathematics but receives little or no attention in text books. The dictionary defines heuristics to be 'the art of discovery' or 'proceeding by trial and error', but I like to think of it as follows:

> *We seek the solution to a problem. If we had that solution, what would it have to look like?*

Thus, to my mind, heuristics in mathematics is akin to psychological profiling in criminal investigations!